HYPOELLIPTICITY OF THE $\bar\partial$-NEUMANN PROBLEM AT EXPONENTIALLY DEGENERATE POINTS

TRAN VU KHANH AND GIUSEPPE ZAMPIERI

ABSTRACT. We prove local hypoellipticity of the complex Laplacian $\Box$ in a domain which has compactness estimates, is of finite type outside a curve transversal to the CR directions and for which the holomorphic tangential derivatives of a defining function are subelliptic multipliers in the sense of Kohn.
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1. Introduction

For the pseudoconvex domain $\Omega \subset \mathbb{C}^n$ whose boundary is defined in coordinates $z = x + iy$ of $\mathbb{C}^n$, by

$$2x_n = \exp\left(-\frac{1}{(\sum_{j=1}^{n-1}|z_j|^2)^{\frac{s}{2}}}\right), \quad s > 0, \tag{1.1}$$

the tangential Kohn Laplacian $\Box_b = \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b$ as well as the full Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ show very interesting features especially in comparison with the "tube domain" whose boundary is defined by

$$2x_n = \exp\left(-\frac{1}{(\sum_{j=1}^{n-1}|x_j|^2)^{\frac{s}{2}}}\right), \quad s > 0. \tag{1.2}$$

(Here z_j have been replaced by x_j at exponent.) Energy estimates are the same for the two domains. For the problem on the boundary $b\Omega$, they come as

$$||(\log\Lambda)^{\frac{1}{s}}u||_{b\Omega} \underset{\sim}{<} ||\bar\partial_b u||^2_{b\Omega} + ||\bar\partial_b^* u||^2_{b\Omega} + ||u||^2_{b\Omega} \tag{1.3}$$

for any smooth compact support form $u \in C_c^\infty(b\Omega)^k$ of degree $k \in [1, n-2]$.

Here $\log\Lambda$ is the tangential pseudodifferential operator with symbol $\log(1 + |\xi'|^2)^{\frac{1}{2}})$, $\xi' \in \mathbb{R}^{2n-1}$, the dual real tangent space. As for the problem on the domain Ω, one has simply to replace $\bar\partial_b$, $\bar\partial_b^*$ by $\bar\partial$, $\bar\partial^*$ and take norms over Ω for forms u in $D_{\bar\partial^*}$, the domain of $\bar\partial^*$, of degree $1 \le k \le n-1$; this can be seen, for instance, in [9]. In particular, these are superlogarithmic (resp. compactness) estimates if $s < 1$ (resp. for

any $s > 0$). A related problem is that of the local hypoellipticity of the Kohn Laplacian $\Box_b$ or, with equivalent terminology, the local regularity of the inverse (modulo harmonics) operator $N_b = \Box_b^{-1}$. Similar is the notion of hypoellipticity of the Laplacian $\Box$ or the regularity of the inverse Neumann operator $N = \Box^{-1}$. It has been proved by Kohn in [12] that superlogarithmic estimates suffice for local hypoellipticity of the problem both in the boundary and in the domain. (Note that hypoellipticity for the domain, [12] Theorem 8.3, is deduced from microlocal hypoellipticity for the boundary, [12] Theorem 7.1, but a direct proof is also available, [7] Theorem 5.4.) In particular, for (1.1) and (1.2), there is local hypoellipticity when $s < 1$.

As for the more delicate hypoellipticity, in the uncertain range of indices $s \geq 1$, only the tangential problem has been studied and the striking conclusion is that the behavior of (1.1) and (1.2) split. The first stays always hypoelliptic for any s (Kohn [11]) whereas the second is not for $s \geq 1$ (Christ [4]). When one tries to relate $(\bar\partial_b, \bar\partial_b^*)$ on $b\Omega$ to $(\bar\partial, \bar\partial^*)$ on Ω, estimates go well through (Kohn [12] Section 8 and Khanh [7] Chapter 4) but not regularity. In particular, the two conclusions about tangential hypoellipticity of $\Box_b$ for (1.1) and non-hypoellipticity for (1.2) when $s \geq 1$, cannot be automatically transferred from $b\Omega$ to Ω. Now, for the non-hypoellipticity in Ω in case of the tube (1.2) we have obtained with Baracco in [1] a result of propagation which is not equivalent but intimately related. The real lines x_j are propagators of holomorphic extendibility from Ω across $b\Omega$. What we prove in the present paper is hypoellipticity in Ω for (1.1) when $s \geq 1$.

Theorem 1.1. *Let Ω be a pseudoconvex domain of $\mathbb{C}^n$ in a neighborhood of $z_o = 0$ and assume that the $\bar\partial$-Neumann problem satisfies the following properties*

(i) *there are local compactness estimates,*
(ii) *there are subelliptic estimates for $(z_1, ..., z_{n-1}) \neq 0$,*
(iii) *$\partial_{z_j} r,\ j = 1, ..., n-1$, are subelliptic multipliers (cf. [10]).*

Then $\Box$ is locally hypoelliptic at z_o.

The proof follows in Section 2. It consists in relating the system on Ω to the tangential system on $b\Omega$ along the guidelines of [12] Section 8, and then in using the argument of [11] simplified by the additional assumption (i).

Remark 1.2. The domain with boundary (1.1), but not (1.2), satisfies the hypotheses of Theorem 1.1 for any $s > 0$: (i) is obvious, and (ii) and (iii) are the content of [11] Section 4.

Notice that $\partial\Omega$ is given only locally in a neighborhodd of z_o. We can continue $\partial\Omega$ leaving it unchanged in a neighborhood of z_o, making it strongly pseudoconvex elsewhere, in such a way that it bounds a relatively compact domain $\Omega\subset\subset\mathbb{C}^n$ (cf. [14]). In this situation $\Box$ is hypoelliptic at every boundary point. Also, it is well defined a H^0 inverse Neumann operator $N=\Box^{-1}$, and, by Theorem 1.1, the $\bar\partial$-Neumann solution operator $\bar\partial^*N$ preserves $C^\infty(\bar\Omega)$-smoothness. It even preserves the exact Sobolev class H^s according to Theorem 2.7 below. In other words, the canonical solution $u=\bar\partial^*Nf$ of $\bar\partial u=f$ for $f\in$ Ker $\bar\partial$ is H^s exactly at the points of $b\Omega$ where f is H^s. The Bergman projection B also preserves $C^\infty(\bar\Omega)$-smoothness on account of Kohn's formula $B=\mathrm{Id}-\bar\partial^*N\bar\partial$.

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2. Hypoellipticity of $\Box$ and exact hypoellipticity of $\bar\partial^*N$

We state properly hypoellipticity and exact hypoellipticity of a general system (P_j).

Definition 2.1. (i) The system (P_j) is locally hypoelliptic at $z_o\in b\Omega$ if

$$P_ju\in C^\infty(\bar\Omega)^k_{z_o}\text{ for any } j \text{ implies } u\in C^\infty(\bar\Omega)^k_{z_o},$$

where $C^\infty(\bar\Omega)^k_{z_o}$ denotes the set of germs of k-forms smooth at z_o.
(ii) The system (P_j) is exactly locally hypoelliptic at $z_o\in b\Omega$ when there is a neighborhood U of z_o such that for any pair of cut-off functions ζ and ζ' in $C^\infty_c(U)$ with $\zeta'|_{\mathrm{supp}(\zeta)}\equiv 1$ we have for any s and for suitable c_s

$$||\zeta u||^2_s\le c_s\Big(\sum_j||\zeta' P_ju||^2_s+||u||^2_0\Big),\qquad u\in C^\infty(\bar\Omega)^k\cap D_{(P_j)}. \tag{2.1}$$

If (P_j) happens to have an inverse, this is said to be locally regular and locally exactly regular in the situation of (i) and (ii) respectively.

Remark 2.2. By Kohn-Nirenberg [13] the assumption $u\in C^\infty$ can be removed from (2.1). Precisely, by the elliptic regularization, one can prove that if $\zeta' P_ju\in H^s$ and $\zeta' u\in H^0$, then $\zeta u\in H^s$ and satisfies (2.1). This motivates the word "exact", that is, Sobolev exact. Not only the local C^∞- but also the H^s-smoothness passes from P_ju to u.

Let ϑ be the formal adjoint of $\bar\partial$ and $\Delta=\bar\partial\vartheta+\vartheta\bar\partial$ the Laplacian; it acts on forms by the action of the usual Laplacian on its coefficients.

If $u \in D_\Box$, then $\Box u = \Delta u$. We first prove exact hypoellipticity of the system $(\bar\partial, \bar\partial^*, \Delta)$; hypoellipticity of $\Box$ itself will follow by the method of Boas-Straube.

Theorem 2.3. *In the situation of Theorem 1.1, we have, for a neighborhood U of z_o and for any couple of cut-off ζ and ζ' with $\zeta'|supp\,\zeta \equiv 1$*

$$||\zeta u||^2_s \underset{\sim}{<} ||\zeta'\bar\partial u||^2_s + ||\zeta'\bar\partial^* u||^2_s + ||\zeta'\Delta u||^2_{s-2} + ||u||^2_0, \quad u \in D_{\bar\partial^*}. \tag{2.2}$$

In particular, the system $(\bar\partial, \bar\partial^*, \Delta)$ is exactly locally hypoelliptic at $z_o = 0$.

Remark 2.4. The hypoellipticity of $\Box_b$ under (ii) and (iii) of Theorem 1.1 is proved by Kohn in [11]. It does not require (i) but it is not exact hypoellipticity (the neighborhood U of (2.1) depends on s). However, inspection of his proof shows that, if (i) is added, then in fact (2.1) holds for $(P_j) = \Box_b$. Our proof consists in a reduction to the tangential system.

Proof. We proceed in several steps which are highlighted in two intermediate propositions. We use the standard notation $Q(u,u)$ for $||\bar\partial u||^2_0 + ||\bar\partial^* u||^2_0$ and some variants as, for an operator Op, $Q_{Op}(u,u) := ||Op\,\bar\partial u||^2_0 + ||Op\,\bar\partial^* u||^2_0$; most often, in our paper, Op is chosen as $\Lambda^s\zeta'$. We decompose a form u as

$$\begin{cases} u = u^\tau + u^\nu, \\ u^\tau = u^{\tau\,+} + u^{\tau\,-} + u^{\tau\,0}, \end{cases}$$

where the first is the decomposition in tangential and normal component and the second is the microlocal decomposition $u^{\tau\,\overset{\pm}{0}} = \Psi^{\overset{\pm}{0}} u^\tau$ in which $\Psi^{\overset{\pm}{0}}$ are the tangential pseudodifferential operators whose symbols $\psi^{\overset{\pm}{0}}$ are a conic decomposition of the unity in the space dual to $\mathbb{R}^{2n-1}$ the real orthogonal to ∂r (cf. Kohn [12]). We begin our proof by remarking that any of the forms $u^\# = u^\nu$, $u^{\tau\,-}$, $u^{\tau\,0}$ enjoys elliptic estimates

$$||\zeta u^\#||^2_s \underset{\sim}{<} ||\zeta'\bar\partial u^\#||^2_{s-1} + ||\zeta'\bar\partial^* u^\#||^2_{s-1} + ||u^\#||^2_0 \qquad s \geq 2. \tag{2.3}$$

We refer to [6] formula (1) of Main theorem as a general reference but also give an outline of the proof. For this, we have to call into play the tangential s-Sobolev norm which is defined by $|||u|||_s = ||\Lambda^s u||_0$. We start from

$$|||\zeta u^\#|||^2_1 \underset{\sim}{<} Q(\zeta u^\#, \zeta u^\#) + ||u^\#||^2_0; \tag{2.4}$$

this is the basic estimate for u^ν (which vanishes at $b\Omega$) whereas it is [12] Lemma 8.6 for $u^{\tau\,-}$ and $u^{\tau\,0}$. Applying (2.4) to $\zeta'\Lambda^{s-1}\zeta u^\#$ one gets the estimate of tangential norms for any s. Finally, by non-characteristicity of $(\bar\partial,\bar\partial^*)$ one passes from tangential to full norms along the guidelines of [16] Theorem 1.9.7. The version of this argument for $\Box$ can be found in [12] second part of p. 245. Because of (2.3), it suffices to prove (2.2) for the only $u^{\tau\,+}$. We further decompose

$$u^{\tau\,+}=u^{\tau\,+\,(h)}+u^{\tau\,+\,(0)},$$

where $u^{\tau\,+\,(h)}$ is the "harmonic extension" in the sense of Kohn [12] and $u^{\tau\,+\,(0)}$ is just the complementary part. We denote by $\bar\partial^\tau$ the extension of $\bar\partial_b$ from $b\Omega$ to Ω which stays tangential to the level surfaces $r\equiv$ const. It acts on tangential forms u^τ and it is defined by $\bar\partial^\tau u^\tau=(\bar\partial u^\tau)^\tau$. We denote by $\bar\partial^{\tau\,*}$ its adjoint; thus $\bar\partial^{\tau\,*}u^\tau=\bar\partial^*(u^\tau)$. We use the notations $\Box^\tau$ and Q^τ for the corresponding Laplacian and energy. We notice that over a tangential form u^τ we have a decomposition

$$Q=Q^\tau+||\bar L_n u^\tau||^2_0. \tag{2.5}$$

The proof of (2.2) for $u^{\tau\,+}$ requires two crucial technical results. Here is the first which is the most central

Proposition 2.5. *For the harmonic extension $u^{\tau\,+\,(h)}$ we have*

$$|||\zeta u^{\tau\,+\,(h)}|||^2_s\underset{\sim}{<} Q^\tau_{\Lambda^s\zeta'}(u^{\tau\,+\,(h)},u^{\tau\,+\,(h)})+||u^{\tau\,+\,(h)}||^2_0. \tag{2.6}$$

Proof. We apply compactness estimates (cf. e.g. [7] Section 6) for $\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}$,

$$||\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}||^2\le\epsilon Q(\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)},\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)})+c_\epsilon||\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}||^2_{-1}. \tag{2.7}$$

We decompose Q according to (2.5). We calculate Q^τ over $\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}$ and compute errors coming from commutators $[Q^\tau,\zeta'\Lambda^s\zeta]$. In this calculation we assume that the cut off functions are of product type $\zeta(z')\zeta(t)$ where z' (resp. t) are complex (resp. totally real) tangential coordinates in $T_{z_o}b\Omega$. We have

$$\begin{aligned}
&Q^\tau(\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)},\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)})\\
&\quad\underset{\sim}{<} {Q^\tau}_{\zeta'\Lambda^s\zeta}(u^{\tau\,+\,(h)},u^{\tau\,+\,(h)})+|||\zeta u^{\tau\,+\,(h)}|||^2_s+|||\zeta' u^{\tau\,+\,(h)}|||^2_{s-1}\\
&\quad+\left(||(|\dot\zeta(z')|+|\dot\zeta'(z')|)\Lambda^s u^{\tau\,+\,(h)}||^2_0+||\sum_{j=1}^{n-1}|r_{z_j}|(|\dot\zeta(t)|+|\dot\zeta'(t)|)\Lambda^s u^{\tau\,+\,(h)}||^2_0\right).
\end{aligned} \tag{2.8}$$

We explain (2.8). First, the commutators $[\bar\partial^\tau, \zeta'\Lambda^s\zeta]$ (and similarly as for $[\bar\partial'^*, \zeta'\Lambda^s\zeta]$) are decomposed by Jacobi identity as

$$[\bar\partial^\tau, \zeta'\Lambda^s\zeta] = [\bar\partial^\tau, \zeta']\Lambda^s\zeta + \zeta'[\bar\partial^\tau, \Lambda^s]\zeta + \zeta'\Lambda^s[\bar\partial^\tau, \zeta].$$

The central commutator $[\bar\partial^\tau, \Lambda^s]$ produces the error term $|||\zeta u^{\tau\,+\,(h)}|||^2_s$. As for the two others, we have

$$[\bar\partial^\tau, \zeta(z')\zeta(t)] = [\bar\partial^\tau, \zeta(z')]\zeta(t) + \zeta(z')[\bar\partial^\tau, \zeta(t)],$$

and similarly for ζ replaced by ζ' and $\bar\partial^\tau$ by $\bar\partial^{\tau\,*}$. Now,

$$[\bar\partial^\tau, \zeta(z')] \sim \dot\zeta(z'). \tag{2.9}$$

On the other hand, we first notice that it is not restrictive to assume that $\partial_{z_1}, ..., \partial_{z_{n-1}}$ are a basis of $T^{1,0}_0 b\Omega$ for otherwise, owing to (iii), we have subelliptic estimates from which local regularity readily follows. Thus, each $\bar L_j,\ j = 1, ..., n-1$, is of type $\bar L_j = r_{\bar z_j}\partial_{\bar z_n} - r_{\bar z_n}\partial_{\bar z_j}$, and then

$$\begin{aligned}[\bar\partial^\tau, \zeta(t)] &\sim \sum_{j=1}^{n-1}[\bar L_j, \zeta(t)]\\ &\sim \sum_{j=1}^{n-1} r_{\bar z_j}\dot\zeta(t).\end{aligned} \tag{2.10}$$

By combining (2.9) with (2.10) (and using the analogous for ζ' and $\bar\partial^{\tau\,*}$), we get the last line of (2.8). This establishes (2.8). Next, since $(\bar\partial^\tau, \bar\partial^{\tau\,*})$ has subelliptic estimates, say η-subelliptic, for $z' \neq 0$ and hence in particular over $\mathrm{supp}\,\dot\zeta(z')$ and $\mathrm{supp}\,\dot\zeta'(z')$ and since the $r_{\bar z_j}$ are, say, η-subelliptic multipliers even at $z' = 0$, then the last line of (2.8) is estimated by $||\zeta''\Lambda^{s-\eta}\zeta' u^{\tau\,+\,(h)}||^2$ where $\zeta'' \equiv 1$ over $\mathrm{supp}\,\zeta'$. This shows, using iteration over increasing k such that $k\eta > s$ and over decreasing j from $s-1$ to 0, that (2.7) and (2.8) imply (2.6) provided that we add on the right side the extra term $||\bar L_n\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}||^2$. Note that, as a result of the inductive process, we have to replace $Q_{\zeta'\Lambda^s\zeta}$ in (2.8) by $Q_{\Lambda^s\zeta'}$ in (2.6).

Up to this point the argument is the same as in [11] and does not make any use of the specific properties of the harmonic extension $u^{\tau\,+\,(h)}$. We start the new part which is dedicated to prove that $||\bar L_n\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}||^2$ can be removed from the right of (2.6). For this we have to use the main property of this extension expressed by [12] Lemma 8.5, that is,

$$||\bar L_n\zeta u^{\tau\,+\,(h)}||^2_0 \underset{\sim}{<} \sum_{j=1}^{n-1}||\bar L_j\zeta u^{\tau\,+}_b||^2_{b,-\frac12} + ||u^{\tau\,+}||^2_0. \tag{2.11}$$

Note that (2.11) differs from [12] Lemma 8.5 by $[\bar L_n, \Psi^+]$; but this is an error term which can be taken care of by $u^{\tau\,0}$ to which elliptic estimates apply. Applying (2.11) to $\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}$ (for the first inequality below), and using the classical inequality $||\cdot||^2_{b,-\frac12}\le c_\epsilon||\cdot||^2_0+\epsilon|||\partial_r\cdot|||^2_{-1}$ (cf. e.g. [8] (1.10)) together with the splitting $\partial_r=\bar L_n+Tan$ (for the second), we get

(2.12)

$$\begin{aligned}
||\bar L_n\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}||^2_0 &\underset{\text{by (2.11)}}{\lesssim} \sum_{j=1}^{n-1}||\bar L_j\zeta'\Lambda^s\zeta u_b^{\tau\,+}||^2_{b,-\frac12}+||\zeta'\Lambda^s\zeta u^{\tau\,+}||^2_0\\
&\lesssim c_\epsilon\sum_{j=1}^{n-1}||\bar L_j\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}||^2_0+\epsilon\sum_{j=1}^{n-1}|||\bar L_n\bar L_j\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}|||^2_{-1}\\
&+\epsilon\sum_{j=1}^{n-1}|||Tan\,\bar L_j\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}|||^2_{-1}+||\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}||^2_0.
\end{aligned}$$

The first term on the right of the last inequality is controlled by $\sum\limits_{j=1}^{n-1}||\zeta'\Lambda^s\zeta\bar L_ju^{\tau\,+\,(h)}||^2+|||\zeta u^{\tau\,+\,(h)}|||^2_s+|||\zeta''u^{\tau\,+\,(h)}|||^2_{s-1}$ by the first part of the proposition; moreover, we have the immediate estimate $\sum_{j=1}^{n-1}||\zeta'\Lambda^s\zeta\bar L_ju^{\tau\,+\,(h)}||^2\lesssim Q^\tau_{\Lambda^s\zeta'}(u^{\tau\,+\,(h)},u^{\tau\,+\,(h)})$. The term which carries ϵTan, after Tan has been annihilated by the Sobolev norm of index -1, has the same estimate as the first term. It remains to control the second term in the right which involves $\epsilon\bar L_n$. We rewrite $\bar L_n\bar L_j=\bar L_j\bar L_n+[\bar L_n,\bar L_j]$; when $\bar L_j$ moves in first position, it is annihilated by -1 and what remains is absorbed in the left. As for the commutator, we have

$$\begin{aligned}
|||[\bar L_n,\bar L_j]\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}|||^2_{-1} &\lesssim |||\zeta u^{\tau\,+\,(h)}|||^2_s+|||\partial_r\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}|||^2_{-1}\\
&\lesssim |||\zeta u^{\tau\,+\,(h)}|||^2_s+|||\bar L_n\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}|||^2_{-1},
\end{aligned}$$

where we have used the splitting $\partial_r=Tan+\bar L_n$ in the second inequality. Again, the term with $\bar L_n$, which now comes in -1 norm, is absorbed in the left of (2.12). Summarizing up, we have got

$$\begin{aligned}
||\bar L_n\zeta'\Lambda^s\zeta u^{\tau\,+\,(h)}||^2_0 &\lesssim c_\epsilon Q^\tau_{\Lambda^s\zeta'}(u^{\tau\,+\,(h)},u^{\tau\,+\,(h)})\\
&+|||\zeta u^{\tau\,+\,(h)}|||^2_s+|||\zeta''u^{\tau\,+\,(h)}|||^2_{s-1}.
\end{aligned}\tag{2.13}$$

But $|||\bar L_n\cdot|||^2$ comes with a factor ϵ of compactness and hence the term in s-norm in the last line can be absorbed in the left of the initial

inequalities (2.7) or (2.6). Finally, we use an inductive argument to go down from $s-1$ to 0. This concludes the proof of the proposition. $\square$

We remark now that

$$\begin{aligned}||\zeta u^{\tau+(h)}||_0^2 &\lesssim ||\zeta u_b^{\tau+}||^2_{b,-\frac12}\\ &\lesssim ||\zeta u^{\tau+}||_0^2+|||\partial_r\zeta u^{\tau+}|||^2_{-1}\\ &\le ||\zeta u^{\tau+}||_0^2+|||\bar L_n\zeta u^{\tau+}|||^2_{-1}+|||Tan\,\zeta u^{\tau+}|||^2_{-1}\\ &\lesssim Q_{\Lambda^{-1}\zeta}(u^{\tau+},u^{\tau+})+||\zeta u^{\tau+}||_0^2.\end{aligned}\tag{2.14}$$

The same inequality also holds for $u^{\tau+(h)}$ replaced by $u^{\tau+(0)}$ on account of the identity $u^{\tau+(0)}=u^{\tau+}+u^{\tau+(h)}$. We need another preparation result

Proposition 2.6. *We have*

$$Q^\tau{}_{\Lambda^s\zeta'}(u^{\tau+(h)},u^{\tau+(h)})\lesssim Q^\tau{}_{\Lambda^s\zeta'}(u^{\tau+},u^{\tau+})+Q^\tau{}_{\partial_r\Lambda^{s-1}\zeta'}(u^{\tau+},u^{\tau+})\tag{2.15}$$

and

$$\begin{aligned}|||\zeta u^{\tau+(0)}|||_s^2\lesssim{}& Q^\tau{}_{\Lambda^{s-1}\zeta'}(u^{\tau+},u^{\tau+})+Q^\tau{}_{\partial_r\Lambda^{s-2}\zeta'}(u^{\tau+},u^{\tau+})\\ &+|||\zeta'\Delta u^{\tau+}|||^2_{s-2}+||u^{\tau+}||_0^2.\end{aligned}\tag{2.16}$$

Proof. The proof of (2.15) is an immediate combination of the formulas $||\zeta' u^{\tau+(h)}||_0\lesssim ||\zeta' u_b^{\tau+}||_{b,-\frac12}$ and $||\zeta' u^{\tau+}||_{b,-\frac12}\lesssim ||\zeta' u^{\tau+}||_0+|||\partial_r\zeta' u^{\tau+}|||^2_{-1}$.

We prove now (2.16). By elliptic estimate for $u^{\tau+(0)}$ (which vanishes at $b\Omega$) with respect to the order 2 elliptic operator Δ, we have

$$|||\zeta u^{\tau+(0)}|||_s^2\lesssim|||\zeta'\Delta u^{\tau+(0)}|||^2_{s-2}+||u^{\tau+(0)}||_0^2.\tag{2.17}$$

This result of Sobolev regularity at the boundary is very classical: it is formulated, for functions in H^1_0 such as the coefficients of $u^{\tau+(0)}$, e.g. in Evans [5] Theorem 5 p. 323. Owing to the identity $\Delta u^{\tau+(0)}=\Delta u^{\tau+}+P^1u^{\tau+(h)}$ for a 1-order operator P^1 (cf. [12] p. 241), we can replace $\Delta u^{\tau+(0)}$ by $\Delta u^{\tau+}$ on the right side of (2.17) putting the contribution of P^1 into an error term of type $|||\zeta' u^{\tau+(h)}|||_{s-1}+|||\zeta'\partial_r u^{\tau+(h)}|||_{s-2}$, which can be estimated, on account of the splitting $\partial_r=\bar L_n+Tan$, by $|||\zeta' u^{\tau+(h)}|||_{s-1}+|||\zeta'' u^{\tau+(h)}|||_{s-2}+Q^\tau{}_{\Lambda^{s-2}\zeta'}(u^{\tau+(h)},u^{\tau+(h)})$. We write the terms of order $s-1$ and $s-2$ as a common $|||\zeta'' u^{\tau+(h)}|||_{s-1}$ that we can estimate, using (2.6) and (2.15), by

$$|||\zeta'' u^{\tau+(h)}|||^2_{s-1}\lesssim Q^\tau_{\Lambda^{s-1}\zeta'''}(u^{\tau+},u^{\tau+})+Q^\tau_{\Lambda^{s-2}\partial_r\zeta'''}(u^{\tau+},u^{\tau+}).$$

This brings down from $s-1$ to 0 the Sobolev index in the error term. This 0-order term $||u^{\tau\,+\,(h)}||^2_0$, together with its companion $||u^{\tau\,+\,(0)}||^2_0$ in the right of (2.17), is estimated, because of (2.14), by $||u^{\tau\,+}||^2_0$ up to a term $Q_{\Lambda^{-1}\zeta}$ which is controlled by the right side of (2.16). This concludes the proof of (2.16).

□

End of proof of Theorem 2.3. We prove (2.2) for $u^{\tau\,+}$; this implies the conclusion in full generality according to the first part of the proof. We have

(2.18)
$$\begin{aligned}|||\zeta u^{\tau\,+\,(h)}|||^2_s \underset{\text{by (2.6)}}{\lesssim} & \; {Q^\tau}_{\Lambda^s\zeta'}(u^{\tau\,+\,(h)},u^{\tau\,+\,(h)})+||u^{\tau\,+\,(h)}||^2_0\\ \underset{\text{by (2.15) and (2.14)}}{\lesssim} & \; {Q^\tau}_{\Lambda^s\zeta'}(u^{\tau\,+},u^{\tau\,+})+{Q^\tau}_{\partial_r\Lambda^{s-1}\zeta'}(u^{\tau\,+},u^{\tau\,+})+||u^{\tau\,+}||^2_0.\end{aligned}$$

We combine (2.18) with (2.16); what we get is

(2.19)
$$\begin{aligned}|||\zeta u^{\tau\,+}|||^2_s &\leq |||\zeta u^{\tau\,+\,(h)}|||^2_s+|||\zeta u^{\tau\,+\,(0)}|||^2_s\\ &\lesssim ||\zeta'\bar\partial u^{\tau\,+}||^2_s+||\zeta'\bar\partial^* u^{\tau\,+}||^2_s+|||\zeta'\Delta u^{\tau\,+}|||^2_{s-2}+||u^{\tau\,+}||^2_0.\end{aligned}$$

By the non-characteristicity of Q, we can replace the tangential norm $|||\cdot|||_s$ by the full norm $||\cdot||_s$ in the left of (2.19). (The explanation of this point can be found, for example, in [12] second part of p. 245.) This proves (2.2) for $u^{\tau\,+}$ and thus also for a general u.

□

We modify $b\Omega$ outside a neighborhood of z_o where it satisfies the hypotheses of Theorem 1.1 so that it is strongly pseudoconvex in the modified portion and bounds a relatively compact domain; in particular, there is well defined the H^0 inverse N of $\Box$ in this domain. There is an immediate crucial consequence of Theorem 2.3.

Theorem 2.7. *We have that*

(2.20) $\bar\partial^* N$ *is exactly regular over* $Ker\bar\partial$

and

(2.21) $\bar\partial N$ *is exactly regular over* $Ker\bar\partial^*$.

Proof. As for (2.20), we put $u = \bar\partial^* Nf$ for $f \in \operatorname{Ker}\bar\partial$. We get

$$\begin{cases} \bar\partial u = f, \\ \bar\partial^* u = 0, \\ \Delta u = (\vartheta\bar\partial + \bar\partial\vartheta)\bar\partial^* Nf \\ \qquad = \vartheta(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)Nf + \bar\partial\vartheta\bar\partial^* Nf \\ \qquad = \vartheta\Box Nf = \vartheta f. \end{cases}$$

Thus, by (2.2)

$$\begin{aligned} ||\zeta u||^2_s &\lesssim ||\zeta' f||^2_s + ||\zeta'\vartheta f||^2_{s-2} + ||u||^2_0 \\ &\lesssim ||\zeta' f||^2_s + ||u||^2_0. \end{aligned} \tag{2.22}$$

To prove (2.21), we put $u = \bar\partial Nf$ for $f \in \operatorname{Ker}\bar\partial^*$. We have a similar calculation as above which leads to the same formula as (2.22) (with the only difference that ϑ is replaced by $\bar\partial$ in the intermediate inequality). Thus from (2.22) applied both for $\bar\partial^* N$ and $\bar\partial N$ on $\operatorname{Ker}\bar\partial$ and $\operatorname{Ker}\bar\partial^*$ respectively, we conclude that these operators are exactly regular.

$\Box$

We are ready for the proof of Theorem 1.1. This follows from Theorem 2.7 by the method of Boas-Straube.

Proof of Theorem 1.1. From the regularity of $\bar\partial^* N$ it follows that the Bergman projection B is also regular. (Notice that exact regularity is perhaps lost by taking $\bar\partial$ in B.) We exploit formula (5.36) in [15] in unweighted norms, that is, for $t = 0$:

$$\begin{aligned} N_q = {} & B_q(N_q\bar\partial)(\mathrm{Id} - B_{q-1})(\bar\partial^* N_q)B_q \\ & + (\mathrm{Id} - B_q)(\bar\partial^* N_{q+1})B_{q+1}(N_{q+1}\bar\partial)(\mathrm{Id} - B_q). \end{aligned}$$

Now, in the right side, the $\bar\partial N$'s and $\bar\partial^* N$'s are evaluated over $\operatorname{Ker}\bar\partial^*$ and $\operatorname{Ker}\bar\partial$ respectively; thus they are exactly regular. The B's are also regular and therefore such is N. This concludes the proof of Theorem 1.1.

$\Box$

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Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy

E-mail address: `khanh@math.unipd.it, zampieri@math.unipd.it`